Lecture 16

In this lecture we continue our study of algorithms.

How do we add fractions?

E.g. $\dfrac{3}{8} + \dfrac{5}{12} = \ ?$

① Oft-used method: $\dfrac{3}{8} + \dfrac{5}{12} = \dfrac{3 \cdot 12}{8 \cdot 12} + \dfrac{5 \cdot 8}{12 \cdot 8}$

$$= \frac{36}{96} + \frac{40}{96}$$

$$= \frac{76}{96} = \frac{38}{48} = \frac{19}{24}$$

② Sensible method: $\dfrac{3}{8} + \dfrac{5}{12} = \dfrac{\boxed{9}}{24} + \dfrac{\boxed{10}}{24} = \dfrac{19}{24}$

Now 96 is the product of 8 and 12.

But what is 24?

It's the <u>least common multiple</u>.
(<u>lowest</u>)

This is what is the best denominator.
(It's the lowest common denominator.)

More *generally*, given integers m and n, their least common multiple (lcm) is the smallest number which is a multiple of them both.

<u>Q^n</u> How do we find it?

<u>Answer</u> For small numbers, trial & error.

In general,

$$\boxed{\operatorname{lcm}(m,n) = \frac{m \cdot n}{\gcd(m,n)}}\ .$$

The least common multiple of 2 positive integers equals their product divided by their greatest common divisor.

We can get the gcd by using the <u>Euclidean algorithm.</u> This involves repeated application of the division algorithm.

Division algorithm

$$a = bq + r$$

quotient remainder

Euclidean algorithm

$$\begin{aligned}
a &= q_0 b + r_1 \\
b &= q_1 r_1 + r_2 \\
r_1 &= q_2 r_2 + r_3 \\
&\vdots \\
r_{n-1} &= q_n r_n + r_{n+1} \\
r_n &= q_{n+1} r_{n+1}
\end{aligned}$$

When the remainder becomes zero, we look back to the previous remainder, r_{n+1}.

This must be the gcd of a and b.

E.g. $\gcd(96, 22) = ?$

$$96 = \underline{4} \cdot 22 + \underline{8}$$
$$22 = \underline{2} \cdot 8 + \underline{6}$$
$$8 = \underline{1} \cdot 6 + \underline{2}$$
$$6 = \underline{3} \cdot 2 \quad \leftarrow \boxed{\text{No remainder.}}$$

The last nonzero remainder was 2.

$$\therefore \gcd(96, 22) = 2$$

$$\therefore \operatorname{lcm}(96, 22) = \frac{96 \cdot \cancel{22}^{11}}{\cancel{2}} = 1056$$

E.g. $\gcd(63, 256) = ?$

$$\cancel{63 = \underline{\quad} \cdot 256 + \underline{\quad}}$$

$$256 = \underline{4} \cdot 63 + \underline{4}$$

$$63 = \underline{15} \cdot 4 + \underline{3}$$

$$4 = \underline{1} \cdot 3 + \underline{1}$$

$$3 = \underline{3} \cdot 1$$ ← No remainder

Last nonzero remainder was 1.

$\therefore \gcd(63, 256) = 1$

$\therefore \operatorname{lcm}(63, 256) = \dfrac{63 \cdot 256}{1} = ?$

Note

We say two positive integers are <u>relatively prime</u> (in relation to each other) if their gcd equals 1.

So 63 and 256 are relatively prime (to each other), even though neither of them is a <u>prime number</u>.

Prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, ...

A <u>prime no.</u> is an integer ≥ 2 which has no factors except itself and 1.